Worksheet for 2020-03-09

Conceptual questions

Question 1. If f is a differentiable function and $\mathbf{u}$ is a unit vector, what is $D_{\mathbf{u}}f(a,b) + D_{-\mathbf{u}}f(a,b)$?

Question 2. If f is a differentiable function defined on all of $\mathbb{R}^2$,

(a) Can f have a local maximum but no absolute maximum?
(b) Can f have an absolute maximum but no local maximum?
(c) Must f have an absolute maximum when constrained to $x + y = 0$? How about $x^4 + y^4 = 7$?
(d) If f has only one critical point, and that point is a local minimum, does that point have to be an absolute minimum?

Question 3. Show that if (a, b) is a critical point for $f(x, y)$ then it is also a critical point for $\phi(f(x, y))$, where f and ϕ are nice and differentiable functions.

Computations

Problem 1. Find and classify the critical point(s) of $f(x, y) = 3xe^y - x^3 - e^{3y}$.

Problem 2. Find and classify the critical point(s) of the function $f(x, y) = \sin x \sin y$.

Problem 3. Consider the function

$$f(x, y) = \int_x^y (6 - t - t^2)\,\mathrm{d}t.$$

(a) Find the critical point(s) of this function. Does it have a absolute maximum?
(b) Find the absolute maximum when the domain is restricted to $y \geq x$.

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. 0

Question 2.

(a) Sure, a function such as $f(x, y) = x^3 - x$ will do.
(b) No: an absolute maximum is by definition also a local maximum.
(c) With a constraint of $x + y = 0$, not necessarily; consider $f(x, y) = x$. With a constraint of $x^4 + y^4 = 7$ then yes by the Extreme Value Theorem.
(d) No. See Exercise 14.7.40 in Stewart.

Question 3. This follows from the chain rule: $\nabla\phi(f(x, y)) = \phi'(f(x, y))\nabla f(x, y)$.

Answers to computations

Problem 1. The only critical point is $(1, 0)$ which by the second derivative test can be classified as a local maximum.

Problem 2. The critical points are

- $(\pi/2 + m\pi, \pi/2 + n\pi)$, which are local maxima when $m + n$ is even and local minima when $m + n$ is odd,
- $(m\pi, n\pi)$, which are saddle points.

Here m, n can be any integers.

Problem 3. You could expand out the integral as a function in terms of x and y, but it's cleaner to not do so and just use the FTC.

(a) The critical points are $(2, 2), (2, -3), (-3, 2), (-3, -3)$. It does not have an absolute maximum: the quantity becomes arbitrarily large as $x \to \infty$ and/or $y \to -\infty$.
(b) This is attained when $x = -3$ and $y = 2$, and the value is $125/6$. (Can you interpret this visually?)